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Anisotropic viscosity of a dispersion of aligned elliptical cylindrical clasts in viscous matrix

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Abstract

Self-consistent averaging, using an auxiliary solution for an elliptical anisotropic viscous inclusion in an anisotropic viscous host, provides estimates of the principal bulk viscosities of a dispersion of aligned elliptical viscous clasts in an isotropic or anisotropic viscous matrix. Analysis and results are for a two-dimensional analog of a composite rock with clasts that are cylinders with axes normal to the plane of flow. The ratio of principal viscosities, η_n in clast parallel extension or shortening, η_s in clast-parallel shear, $m = \eta_n/\eta_s$, is smaller than that obtained using an auxiliary solution in which the host is isotropic. Results for the limiting case of rigid clasts indicates that the latter procedure overestimates the stress concentration in axis-parallel extension or shortening at intermediate clast volume fraction, f. If the matrix is anisotropic, bulk anisotropy derives from both the shape anisotropy and the intrinsic anisotropy of the matrix, and unsymmetrical relations for the principal viscosities and $m(f)$ result. The results suggest that rheological anisotropy in rocks with a planar fabric is greatly reduced if the components are lenticular in form rather than continuous layers. A general solution is given for an elliptical inclusion for the case that the principal axes of anisotropy in both the host and the inclusion are oblique to the axes of the elliptical section and the host is subjected to homogeneous stress far from the inclusion.

Keywords: Anisotropic viscous composite; Elliptical inclusion; Constitutive relations; Rheological anisotropy; Principal bulk viscosities

1. Introduction

Rheological constitutive relations for composite rocks for a flow whose length scale is much greater than that of its mechanically distinct components must generally account for anisotropy. Bulk constitutive relations for layered rock may be readily written down (Treagus, 1993), given the relations for the components. Application of the traction continuity conditions on planar interfaces and the conditions of coherency for the rate of deformation components determine this result. For less regular composite geometry, as for a deformed conglomerate with one or more clast components in matrix (Treagus, 2002, 2003; Treagus and Treagus, 2002), another method is required. These authors treat the components as isotropic linear viscous fluids and employ self-consistent averaging (SCA; Hill, 1965; Treagus, 2002) to obtain estimates of bulk and component properties of clast and matrix materials in a naturally deformed conglomerate. This is a promising method of pursuing the question re-iterated by Talbot (1999): "Can field data constrain rock viscosities?"

This model is useful in treating any configuration of several components if the effective geometry of each component may be represented by a single aspect ratio. The model ignores the contribution to the bulk anisotropy from the intrinsic anisotropy of a component. In this paper, a two-dimensional case is treated, in which the conglomerate contains cylindrical clasts of elliptical cross-section, and is subjected to plane deformation in the section normal to the clast axes.

An example serves to motivate the problem. Given the ratio of principal viscosities, $m = \eta_n/\eta_s$, where η_n is the principal viscosity in lamination-parallel shortening or extension and η_s that in lamination-parallel shear, we may compute the evolution of structures in anisotropic rocks (e.g. Bayly, 1964; Cobbold et al., 1971). Low-slope amplification of a sinusoidal perturbation with amplitude uniform along its axial plane in shortening parallel to the

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Nomenclature

A	fold amplitude
a/b	major to minor semi-axis of elliptical clast section
D_{xx}, D_{xy}	rate of deformation components
$D_{xx}^{(i)}$, $D_{xy}^{(i)}$	rate of deformation components in two mechanical components, $i = 1, 2$
D_{xx}^{*}, D_{xx}^{o}	rate of deformation components in inclusion and far-field
f	volume (area) fraction of clasts
f_1, f_2	volume (area) fraction of two viscous components
m	ratio of bulk principal viscosities
m_U	upper limit of ratio of bulk principal viscosities, for bilaminate
m_1	ratio of principal viscosities of anisotropic matrix
$N(f)$, $N'(f)$	condensed notation for functions of f
R	ratio of viscosities of clasts and matrix in isotropic/isotropic case
R_G	ratio of clast viscosity to geometric mean viscosity of anisotropic matrix
s_{xx}, s_{xy}	deviatoric stress components
$s_{xx}^{(i)}$, $s_{xy}^{(i)}$	deviatoric stress components in components, $i = 1, 2$
s_{xx}^{*}, s_{xx}^{o}	deviatoric stress components in inclusion and far-field
t	time
x, y	coordinates in plane of flow
β, β_n, β_s	ratio of principal viscosity to matrix viscosity
η_1	viscosity of isotropic matrix
η_2	viscosity of isotropic clast
η_n, η_s	principal normal and shear viscosities of composite
$\eta_n^{(1)}$, $\eta_s^{(1)}$	bulk principal viscosities in matrix
η^{*}, η_n^{*}, η_s^{*}	bulk viscosities in inclusion
ν	shape factor, function of a/b
σ, σ^{*}, σ^{o}	mean in-plane normal stress, in inclusion, in far-field
σ_{xx}, σ_{yy}, σ_{xy}	stress components
$\sigma_{xx}^{(i)}$, $\sigma_{yy}^{(i)}$, $\sigma_{xy}^{(i)}$	stress components in component materials, $i = 1, 2$
σ_{yy}^{*}, σ_{yy}^{o}	stress components in inclusion & far-field
θ	function of ν and f

stiff direction in a homogeneous anisotropic viscous fluid is given by

$$\frac{dA}{dt} = -[4(m-1)+1]\bar{D}_{xx}A \tag{1}$$

where A is amplitude and $\bar{D}_{xx}$ is the basic-state rate of shortening (Bayly, 1964; Fletcher and Pollard, 1999). For a viscous bilaminate (Treagus, 2002) composed of alternating layers of two isotropic viscous fluids with viscosities η_1 and η_2, in proportions f_1 and $f_2 = 1 - f_1$

$$m = m_U = \left(\frac{f_1}{\eta_1} + \frac{f_2}{\eta_2}\right)(f_1\eta_1 + f_2\eta_2) = (1-f)^2 + f(1-f)\left(R + \frac{1}{R}\right) + f^2 \tag{2}$$

where $f = f_2$ and $R = \eta_2/\eta_1$. The bilaminate has maximal anisotropy, or maximum m, for a composite, and m_U denotes an upper bound. Strong folding instability requires the term in brackets to be >20–30, or $m > 4$–5; instability is weak if it is <5–10, or $m < 2$–3. If the mechanical components have clast-like or lenticular form, as in conglomerates or gneisses, the ideal bilaminate may not afford a good approximation. Treagus (2003) has estimated the principal bulk viscosities of a dispersion of aligned elliptical cylindrical clasts in a matrix, when both are isotropic viscous fluids. The ratio ranges from $m = 1$ for clast aspect ratio $a/b = 1$ (Fig. 1) to $m = m_U$, for $a/b \gg 1$.

Treagus (2003) uses SCA to estimate the bulk properties of a dispersion of aligned elliptical isotropic viscous clasts in an isotropic viscous matrix using the auxiliary solution for an elliptical inclusion in an isotropic matrix. In this paper, use of a solution for an inclusion in an anisotropic host yields substantially lower estimates for m. The difference will affect interpretation of observations of structures such as folds in anisotropic rock. Mandal et al. (2000) estimate the bulk properties of a dispersion of aligned ellipsoidal inclusions of viscous fluid in a viscous matrix. Their method is significantly different from that used here, and no attempt at a comparison between the

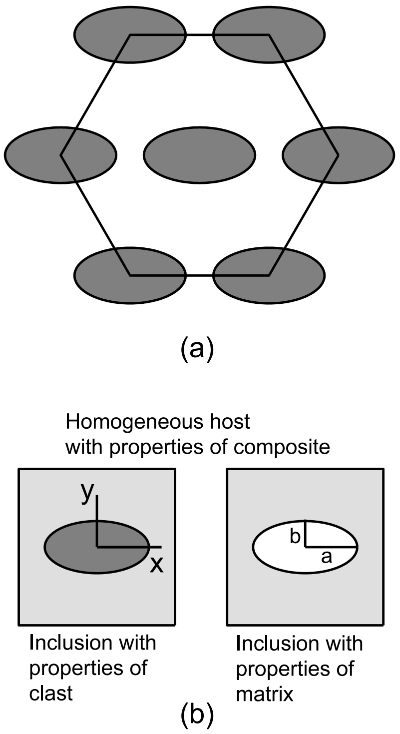

Fig. 1. Schematic of composite made up of aligned elliptical clasts in matrix. (a) Dispersion of aligned elliptical cylindrical clasts of component 2 in matrix of component 1. Clasts, with aspect ratio $a/b = 2$, are positioned on a regular hexagonal grid; the area fraction of clasts is $f \cong 0.30$. The formulation is not restricted to a regular array: clasts may have random position and size, but must have equal aspect ratio. (b) Inclusion-in-host configurations for the auxiliary problem for clast and matrix components; the homogeneous host is light gray to suggest intermediate properties.

formulation and results of the present method and theirs has been made.

2. Self-consistent averaging and Treagus' (2003) estimates

Self-consistent averaging to obtain the bulk properties of a composite material has two steps. First, the components of stress and rate of deformation in a representative element of each component is obtained by some means. In a common method, used here, the representative element is modeled as an inclusion embedded in a homogeneous material with the postulated bulk properties of the composite. A solution to this auxiliary problem may be found if the inclusion has ellipsoidal or elliptical cylindrical shape. The bulk properties enter the solution, and these are estimated by averaging the components of stress or rate of deformation over the volume distribution of the components and setting this equal to the applied value. The term component is used here in two distinct but familiar ways. Averaging leads to nonlinear, generally coupled, algebraic equations in the bulk properties, whose order is equal to the number of components.

To fix ideas, consider a bilaminate made up of two isotropic viscous components. The inclusion in this case is a single layer of one of the component fluids. We postulate that the bulk constitutive relations, referred to as axes parallel and normal to layering and specialized for plane flow, are

$$D_{xx} = \frac{1}{2\eta_n} s_{xx} \qquad D_{xy} = \frac{1}{2\eta_s} s_{xy} \tag{3}$$

where $s_{xx} = 1/2(\sigma_{xx} - \sigma_{yy})$ and $s_{xy} = \sigma_{xy}$ are the deviatoric stress components in plane flow in the x-, y-plane. Let a layer with viscosity η_1 be embedded in the medium parallel to the principal axis x. Then

$$D_{xx} = D_{xx}^{(1)} \qquad \frac{1}{2\eta_n} s_{xx} = \frac{1}{2\eta_n^{(1)}} s_{xy}^{(1)} \qquad s_{xx}^{(1)} = \frac{\eta_n^{(1)}}{\eta_n} s_{xx} \tag{4}$$

Since $\sigma_{yy}^{(1)} = \sigma_{yy}^{(2)} = \sigma_{yy}$, averaging σ_{xx} yields

$$f_1 s_{xx}^{(1)} + f_2 s_{xx}^{(2)} = s_{xx} \tag{5}$$

From Eq. (4), its equivalent for $s_{xx}^{(2)}$, and Eq. (5), we obtain the estimate

$$\eta_n = f_1\eta_1 + f_2\eta_2 \tag{6}$$

Since, for the embedded layers

$$s_{xy}^{(1)} = s_{xy}^{(2)} = s_{xy} \tag{7}$$

it is necessary to average the component of the rate of deformation in the development

$$f_1 D_{xy}^{(1)} + f_2 D_{xy}^{(2)} = D_{xy} \qquad f_1\left(\frac{s_{xy}}{2\eta_1}\right) + f_2\left(\frac{s_{xy}}{2\eta_2}\right) = \frac{s_{xy}}{2\eta_s}$$
$$\eta_s = \left(\frac{f_1}{\eta_1} + \frac{f_2}{\eta_2}\right)^{-1} \tag{8}$$

The present exact results may be derived without mentioning SCA, yielding Eq. (2).

Treagus (2003) notes that these estimates are symmetric under interchange of indices 1 and 2, since both components have the same geometric form. Treagus (2003) calls the ratio m (her δ), also symmetric under interchange of indices, the anisotropy factor.

3. New auxiliary solution and further development of estimates

In Treagus' (2003) estimate for the principal viscosities, η_n and η_s, of a dispersion of aligned cylindrical inclusions, or clasts, with elliptical cross-section embedded in matrix, the auxiliary solution used is that for an isotropic inclusion in an isotropic medium. Thus, a question as to the self-consistency of the method arises. On the other hand, SCA might be said to have only the following three requirements. (i) Estimates for the average stress and rate of deformation components are obtained by any plausible means. (ii) The volume average of a Cartesian tensor component should equal its bulk value. (iii) Estimates of bulk properties derived by averaging stress components should equal those obtained by averaging rate of deformation components. The auxiliary problem used by Treagus (2003) may be viewed as the best available. Now another is given.

A solution (Appendix A) for an elliptical inclusion with principal viscosities η_n^* and η_s^* embedded in a host with principal viscosities η_n and η_s, with the inclusion aligned in a direction of maximum stiffness common to host and inclusion, gives inclusion stress components

$$\sigma^* = \sigma^o + \left[\frac{\left(\frac{a}{b}-\frac{b}{a}\right)}{\left(\frac{a}{b}+\frac{b}{a}\right)}\right](s_{xx}^* - s_{xx}^o)$$

$$s_{xx}^* = \left[\frac{(\sqrt{m}+\nu)}{\left(\sqrt{m}+\nu\frac{\eta_n}{\eta_n^*}\right)}\right]s_{xx}^o \tag{9}$$

$$s_{xy}^* = \left[\frac{(\sqrt{m}+\nu)}{\left(\sqrt{m}\frac{\eta_s}{\eta_s^*}+\nu\right)}\right]s_{xy}^o$$

where

$$\sigma = \frac{1}{2}\left(\sigma_{xx}+\sigma_{yy}\right) \tag{10}$$

σ^* and σ^o are inclusion and far-field host quantities, respectively, and

$$m = \frac{\eta_n}{\eta_s} \qquad \nu = \frac{1}{2}\left(\frac{a}{b}+\frac{b}{a}\right) \tag{11}$$

If the inclusion is circular, $\nu = 1$, and the host is isotropic, $m = 1$, Eq. (11) reduces to

$$\sigma^* = \sigma^o \qquad s_{xx}^* = \left[\frac{2\eta_n^*}{(\eta_n^*+\eta)}\right]s_{xx}^o$$
$$s_{xy}^* = \left[\frac{2\eta_s^*}{(\eta_s^*+\eta)}\right]s_{xy}^o \tag{12}$$

These amend the result for an isotropic inclusion by an intuitively reasonable dependence on η_n^* and η_s^*.

The relations (9) are more complicated than those used by Treagus (2003) because both the inclusion and the host material are anisotropic. For isotropic host and inclusion, Eq. (9) reduces to relations equivalent to those used by her, with

$$s_{xx}^* = \left[\frac{(1+\nu)}{\left(1+\nu\frac{\eta}{\eta^*}\right)}\right]s_{xx}^o \qquad s_{xy}^* = \left[\frac{(1+\nu)}{\left(\frac{\eta}{\eta^*}+\nu\right)}\right]s_{xy}^o \tag{13}$$

ν is her shape factor p; η^* applies to the inclusion and η to the host.

Now apply SCA to s_{xx}, using Eq. (13). Suppose the dispersed cylindrical clasts have the same aspect ratio, so a single value of ν applies. The relation (13) gives $s_{xx}^{(2)}$ when η^* is replaced by η_2. To implement SCA, we require an estimate for the matrix component with viscosity η_1. A somewhat counter-intuitive step is taken in which the matrix component is represented by an inclusion with the same aspect ratio as the clasts (Treagus, 2003; Fig. 1). It was initially felt that the interstitial matrix component might be better approximated by an equant inclusion, $\nu = 1$, but as may be shown, unless the representative inclusion for both components has the same shape, the estimate obtained by averaging s_{xx} will differ from that obtained by averaging D_{xx}, and the condition (iii) will not be satisfied. Using the same aspect ratio $\nu_1 = \nu_2 = \nu$, and applying SCA, or

$$(1-f)s_{xx}^{(1)} + fs_{xx}^{(2)} = s_{xx} \tag{14}$$

the first expression in Eq. (13) gives

$$\nu\beta^2 + \{[f+(1-f)R] - \nu[(1-f)+fR]\}\beta - R = 0 \tag{15}$$

where $\beta = \eta/\eta_1$, $R = \eta_2/\eta_1$ and $f = f_2$.

Repeating the procedure for s_{xy}

$$\beta^2 + \{\nu[f+(1-f)R] - [(1-f)+fR]\}\beta - \nu R = 0 \tag{16}$$

The relations (15) and (16) do not give the same values for β; Eq. (16) may be obtained from Eq. (15) by replacing ν by $1/\nu$ (Treagus, 2003).

Because Eq. (15) may also be obtained from

$$(1-f)D_{xx}^{(1)} + fD_{xx}^{(2)} = D_{xx} \tag{17}$$

Treagus (2003) identifies its solution as $\beta_n = \eta_n/\eta_1$, since η_n is the bulk viscosity in clast axis parallel extension or shortening. She then evokes the symmetry between η_n/η_1 and η_s/η_1 for the bilaminate to identify the solution for $\nu \to 1/\nu$, or for Eq. (16), with the ratio $\beta_s = \eta_s/\eta_1$. More directly, we associate the value of β obtained for s_{xy} or D_{xy} with η_s/η_1.

This estimate has two difficulties. First, estimates for two principal bulk viscosities are obtained using an auxiliary problem with an isotropic host rather than one with the effective anisotropic properties of the composite. The association of the estimates for the viscosity of the isotropic material, η, with those for the expected principal viscosities, η_n and η_s, has an ad hoc element. The difficulty is more clearly seen when the orientation of the long axes of the clasts are preferentially, but not perfectly, aligned. A fixed orientation of the principal axes of anisotropy for the bulk material can no longer be identified in an isotropic host, so that estimation of bulk properties cannot be carried out. Second, the estimates obtained must be different from those using the solution for an anisotropic host, although the difference may not be quantitatively significant. We will now discover whether this is so.

Self-consistent averaging based on Eq. (9) for two isotropic components leads to the relations

$$(\sqrt{m}+\nu)\left[\frac{f}{\left(\nu\frac{\beta_n}{R}+\sqrt{m}\right)}+\frac{(1-f)}{(\nu\beta_n+\sqrt{m})}\right]=1$$

$$(\sqrt{m}+\nu)\left[\frac{f}{\left(\nu+\frac{\beta_n}{R\sqrt{m}}\right)}+\frac{(1-f)}{\left(\nu+\frac{\beta_n}{\sqrt{m}}\right)}\right]=1 \quad (18)$$

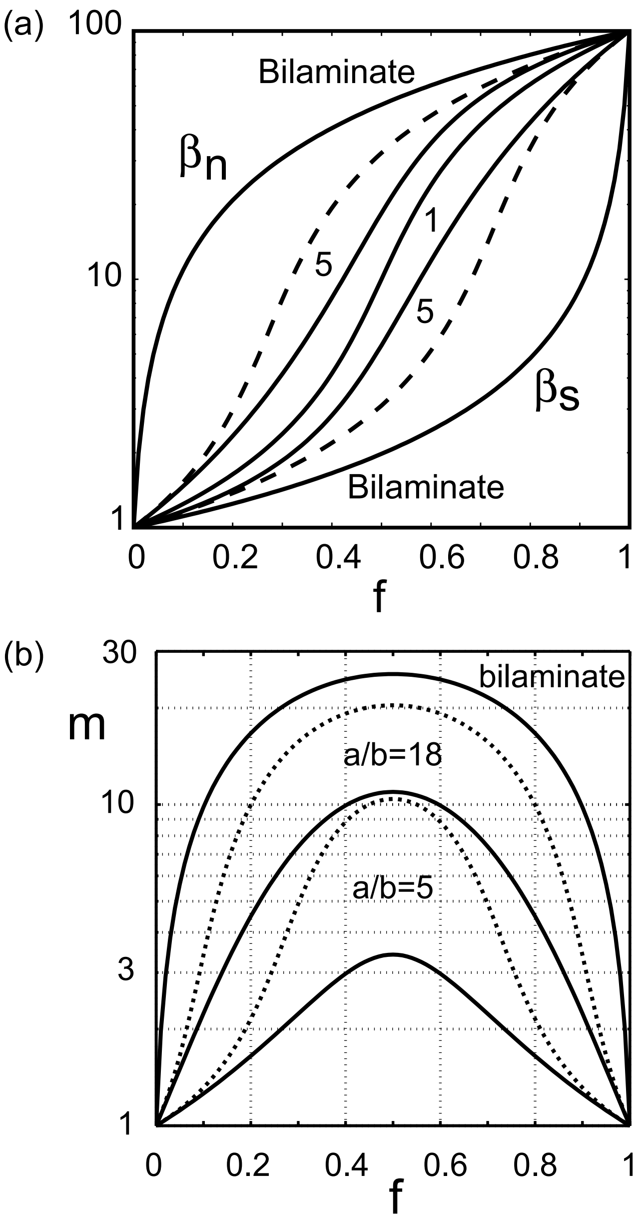

Fig. 2. (a) Principal normalized viscosities $\beta_n = \eta_n/\eta_1$ and $\beta_s = \eta_s/\eta_1$ for aspect ratios $a/b = 1$, 5, and ∞; (b) ratios $m = \eta_n/\eta_s$ for $a/b = 5$, 18, and ∞. Dashed lines are estimates using Treagus' method; solid lines are the results for the present method and the bilaminate limits.

where β_n and m are both unknown quantities, and a simultaneous solution of the two equations is required. Eq. (18) may be recast into the form of Eqs. (15) and (16)

$$\nu\beta_n^2+\left\{\sqrt{m}\left[f+(1-f)R\right]-\nu\left[(1-f)+fR\right]\right\}\beta_n-\sqrt{m}R=0$$

$$\beta_n^2+\left\{\nu\sqrt{m}\left[f+(1-f)R\right]-m\left[(1-f)+fR\right]\right\}\beta_n-\nu m\sqrt{m}R=0 \quad (19)$$

These equations are not symmetric under interchange of ν and $1/\nu$. To solve for given values of f, ν and R, we substitute a closely spaced set of values of m in the first equation and solve it to obtain a closely spaced set of β_n. From these, the pair is chosen which minimizes the absolute value of the left-hand side of the second equation, or is also a solution to it.

Results for η_n/η_1, η_s/η_1 and $m = \eta_n/\eta_s$ obtained by Treagus' method and the present method for $R = \eta_2/\eta_1 = 100$ are compared in Fig. 2a for $a/b = 1$, 5 and ∞, and in Fig. 2b for $a/b = 5$, 18 and ∞. Compare figures 4 and 5 in Treagus (2003). The present method gives a markedly lower bulk anisotropy. For small or large clast fractions, $f < 0.15$ or $f > 0.80$, the results are nearly coincident; for intermediate fractions the divergence is large and maximum at $f = 0.5$.

4. Dispersions of rigid clasts in an isotropic viscous matrix

To seek insight into the difference between the present estimates and those of Treagus (2003), consider aligned elliptical rigid inclusions in an isotropic matrix. For this case, the differences are greatest, and simple closed-form solutions exist. Because viscosity estimates become infinite for critical values of f, and because the forms of the curves are generally simpler, we give results in terms of the *reciprocals* of the normalized principal viscosities, $1/\beta_n$ and $1/\beta_s$, and the ratio $m = \beta_n/\beta_s$. For circular rigid inclusions, the familiar result is

$$\frac{1}{\beta_n}=\frac{1}{\beta_s}=\frac{1}{\beta}=1-2f \quad (20)$$

This is a straight line in a plot of $1/\beta$ versus f, with the viscosity estimate taking an infinite value at $f = 0.5$. The negative values of $1/\beta$ for larger f have no physical meaning. Circular inclusions of equal radius packed in a square array have $f = \pi/4 = 0.78$ and in a hexagonal closed-packed array have a maximum value $f = \pi/(2\sqrt{3}) = 0.91$. Since the dispersions need not have clasts of equal size, a limiting value of $f = 1$ may be approached. Thus, the critical value of f at which the composite becomes rigid is far less than values at which the clasts are separated by ample thicknesses of viscous fluid to allow bulk deformation. For circular clasts in a square array, the separation between clasts at $f = 0.5$ is one half their radius. Certainly, there is no reason to suppose that the effective viscosity of such an array is infinite!

Applying SCA to the rigid inclusion case, we find the simple, symmetric relations

$$\frac{1}{\beta_n} = 1 - f\left(1 + \frac{\nu}{\sqrt{m}}\right) \qquad \frac{1}{\beta_s} = 1 - f\left(1 + \frac{\sqrt{m}}{\nu}\right) \tag{21}$$

Substitution of Eq. (21) into

$$\frac{1}{\beta_n} = \frac{m}{\beta_s} \tag{22}$$

yields a quadratic equation in $\sqrt{m}$, whose solution is

$$\sqrt{m} = \theta + \sqrt{\theta^2 + 1} \qquad \theta = \frac{\left(\nu f - \frac{f}{\nu}\right)}{2(1-f)} \tag{23}$$

Using values of m from Eq. (23) in Eq. (21) yields the desired results.

The Treagus estimates are obtained by replacing $\sqrt{m}$ in Eq. (21) by one, yielding

$$\frac{1}{\beta_n} = 1 - f(1+\nu) \qquad \frac{1}{\beta_s} = 1 - f\left(1 + \frac{1}{\nu}\right) \tag{24}$$

from which

$$m = \frac{1 - f\left(1 + \frac{1}{\nu}\right)}{1 - f(1+\nu)} \tag{25}$$

A plot of the reciprocals of the normalized principal viscosities versus rigid inclusion fraction f, for $a/b = 5$ as computed by the two methods (Fig. 3a) illustrates their similarities and differences. Both give nearly the same results at small $f \approx 0.10$–0.15. The difference is greatest near the critical fractions at which the estimated composite viscosities become infinite. The present method yields, surprisingly, a single fixed value at which this occurs, $f = 0.5$, independent of aspect ratio, a/b. Treagus' method gives two critical values, at the lower of which η_n becomes infinite ($1/\beta_n \to 0$) and at the upper of which η_s becomes infinite ($1/\beta_s \to 0$); both shift with a/b. Relative to that of the present method, this result may be viewed as a *disproportionation*, since the two critical values lie on both sides of $f = 0.5$. This behavior also applies to the viscous-rigid bilaminate limit, for which the critical values are $f = 0+$ and 1. The Treagus result approaches the bilaminate limits as a/b increases, which may be viewed as an argument in its favor.

Alternatively, the rigid-viscous bilaminate limit may be considered to impose too strong a mechanical constraint on bulk deformation of a dispersion of inclusions of finite dimension. Existence of two critical values in the Treagus estimate has the following consequences. When the lower critical value is approached at $f = \nu/(1+\nu)$, from Eq. (24), or, for $a/b = 5$, at $f = 0.28$, η_n/η_1 rises sharply towards an infinite value. The composite is rigid in clast axis parallel shortening or extension, but may still deform in axis-parallel shear: the behavior of a rigid-viscous bilaminate. At the higher critical value, $f = 1/(1+\nu)$, or for the present case, $f = 0.72$, the composite becomes perfectly rigid. If these estimates were reliable, a composite in which elliptical nearly rigid clasts were embedded in a viscous matrix would exhibit remarkably large anisotropy.

The anisotropy factor m (Fig. 3b) varies smoothly as a function of a/b through and beyond the critical value $f = 0.5$. However, at $f = 0.5$, $m = 0/0$ is indeterminate, and the smooth variation beyond is obtained from a ratio of *negative* reciprocal viscosities and, hence, meaningless. For $a/b = 5$, the maximum value reached at $f = 0.5$ is the finite value $m = 6.8$.

The present results, in which the bulk anisotropic properties of the composite are explicitly incorporated, indicate that once the full integrity of a laminated material is broken, as by division into a dispersion of elliptical clasts, even of large aspect ratio, a much less constrained kinematics at the microscopic scale sets in. Flow involves both the principal normal and shear viscosities. A partial approach to the bilaminate limit does take place as a/b increases, although the critical fraction $f = 0.5$ continues to hold (Fig. 3c). As a/b increases, a larger and larger range in $f < 0.5$ exhibits finite, but very large, viscosity in axis parallel shortening or extension. Very large values of a/b are required to approach the bilaminate limit for $1/\beta_n$. On the other hand, the normalized shear viscosity rapidly goes to the curved locus that terminates at $f = 0.5$, rather than $f = 1$, as in the bilaminate limit.

Since the reciprocal of both principal viscosities go to zero at $f = 0.5$, substitution into Eq. (25) gives

$$m(0.5) = \nu^2 \tag{26}$$

This provides an upper estimate on m for any a/b and $f < 0.5$, as well as for clasts with finite viscosity in an isotropic matrix. By effective reduction to a viscous rigid bilaminate at the critical f at which β_n becomes infinite, the Treagus method gives a limit $m \to \infty$ for a dispersion of rigid inequant clasts, except for the case $a/b = 1$.

A simple explanation can be given for the critical values of f at which estimated viscosities become infinite. For a circular rigid inclusion in an isotropic matrix, from Eq. (13), the stress concentration is two: the deviatoric stress in the inclusion is twice the far field value. Thus, in contributing to the average, 50% rigid component 'uses up' the far field stress component, the deviatoric stress in the deformable matrix must be zero or negative, and the composite cannot deform. The stress concentration for the normal component of the deviatoric stress in the elliptical rigid inclusion in an isotropic host is $(1+\nu)/\nu$ so that at $f = \nu/(1+\nu)$ the rigid clasts support the entire far-field stress component, s^o_{xx}. We

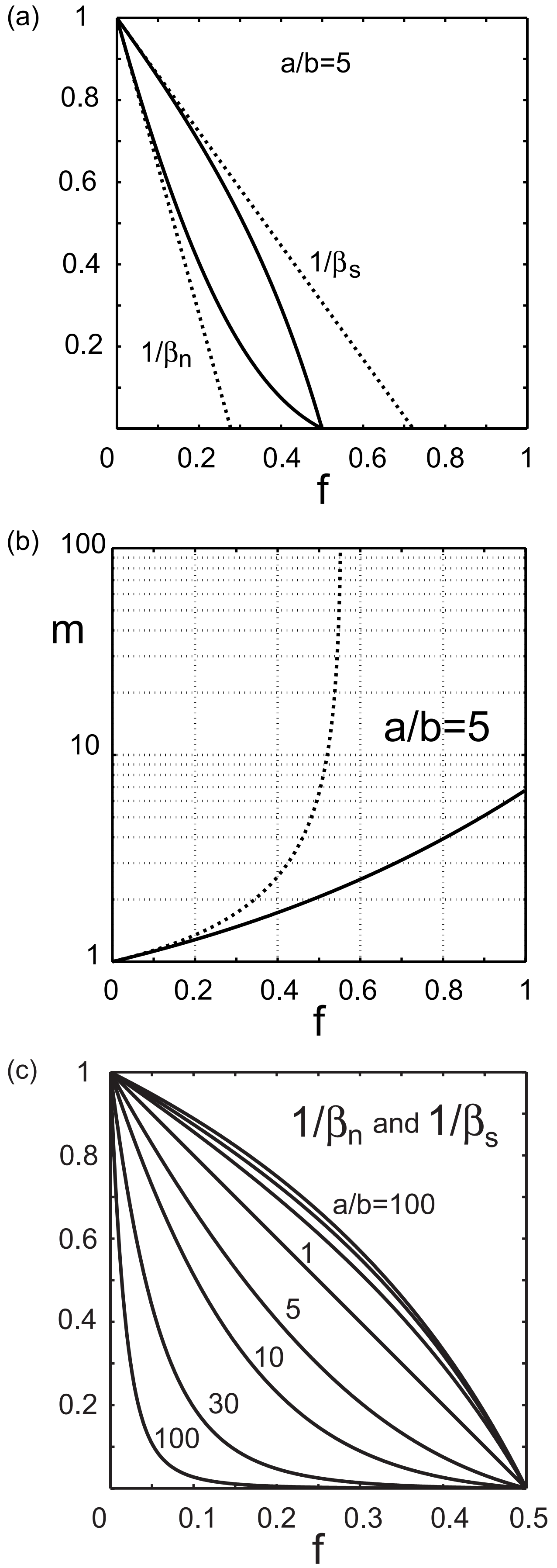

Fig. 3. Reciprocals of normalized principal viscosities (a), and their ratio $m = \eta_n/\eta_s$ (b), for the case $a/b = 5$. Dashed lines are obtained by Treagus' method. (c) Reciprocals of normal and shear viscosity obtained by present method for $a/b = 1$, 5, 10, 30 and 100. The three curves on the right are for $1/\beta_n$ for $a/b = 5$, 10 and 100.

conclude that, if the host is anisotropic, the stress concentrations for both s^*_{xx} and s^*_{xy} obtained from the auxiliary solution for $f = 0.5$ are two, independent of the aspect ratio, a/b. From Eq. (9), with $\eta^*_n \to \infty$ and $\eta^*_s \to \infty$

$$s^*_{xx} = \left[\frac{(\sqrt{m} + \nu)}{\sqrt{m}} \right] s^o_{xx} \qquad s^*_{xy} = \left[\frac{(\sqrt{m} + \nu)}{\nu} \right] s^o_{xy} \tag{27}$$

The relation (26) gives $\sqrt{m} = \nu$, and both coefficients do indeed reduce to two at $f = 0.5$. At smaller f, $\sqrt{m} < \nu$, and so the stress concentration is larger in s^*_{xx}, i.e. $1 + \nu/\sqrt{m}$, and smaller in s^*_{xy}, i.e. $1 + \sqrt{m}/\nu$. In the former case, the product of the stress concentration and f must be less than unity for all $f < 0.5$. Note that the value of m is not determined solely by the auxiliary problem, but arises from application of SCA.

It is clear that material consisting of a dispersion of rigid circular clasts of fraction $f = 0.5$ in a viscous fluid can deform. The proper conclusion is that the estimate of the stress concentration at two is not a good one. One might get a better estimate from a numerical model or from other simple means (Paul, 1960). Treagus' (2003) method of estimation, using an elliptical inclusion in an isotropic medium, gives a stress concentration on axis-parallel or normal shortening of $1 + 1/2(a/b + b/a)$, which reduces to two for $a/b = 1$, but is otherwise larger. Hence, the volume fraction at which the rigid component 'uses up' all the available deviatoric stress for this component is the reciprocal of this quantity, and the critical value of f is less than 0.5. Again, the stress concentration is over-estimated.

Three features of the present estimate suggest it is an improvement over the Treagus estimate. (i) Estimates of the stress concentration for rigid clasts by both methods, and by extension, to deformable clasts, are too large for non-dilute dispersions. Smaller estimates obtained from the present method are more realistic. (ii) Estimated onset of rigidity in both shear and shortening/extension parallel to clast axes is more realistic in dispersions than a sudden onset of infinite value in η_n alone. (iii) Self-consistency achieved by treating the host as anisotropic in the auxiliary problem should result in a better estimate.

5. Symmetry in η_n and η_s

Treagus (2002) notices that curves for log (η_n/η_1) and log (η_s/η_1) as functions of volume fraction f, are related through a center of symmetry at $f = 1/2$ and (1/2)log (R), in our notation, for the bilaminate and for a dispersion of circular clasts. She infers that the same symmetry applies to a dispersion of aligned elliptical inclusions and uses this to obtain estimates for both principal viscosities.

For convenience denote these two curves as $N(f) = \log(\eta_n/\eta_1)$ and $N'(f) = \log(\eta_s/\eta_1)$. The symmetry means that any straight line through the center of symmetry that cuts one of the curves will cut both at points equidistant from it. In the isotropic case, a single curve is cut twice by lines of limited orientation through the center of symmetry. For such pairs of points, since their f-coordinates, f and $1-f$, will be equally-spaced from the center of symmetry, $f = 1/2$, this condition requires that their vertical distance from the line (1/2)log (R) be equal or

$$N(1-f) - \tfrac{1}{2}\log(R) = \tfrac{1}{2}\log(R) - N'(f)$$
$$N(f) - \tfrac{1}{2}\log(R) = \tfrac{1}{2}\log(R) - N'(1-f) \tag{28}$$

or

$$N(1-f) + N'(f) = N'(1-f) + N(f) = \log(R) \tag{29}$$

From the definition of the ratio of bulk viscosities

$$N(f) - N'(f) == \log(m) \tag{30}$$

Re-arranging Eq. (29)

$$N(1-f) - N'(1-f) = N(f) - N'(f) \tag{31}$$

Thus, a necessary condition for a center of symmetry is that

$$m(1-f) = m(f) \tag{32}$$

Indeed, this applies for the bilaminate. The relation (32) and the condition that the curves for η_n and η_s intersect at $f = 0$ and 1 are necessary and sufficient conditions for the center of symmetry, yielding Eq. (28).

Treagus postulates that the N-curves providing estimates for η_n/η_1 for intermediate aspect ratio materials may be augmented by N'-curves by this operation, providing estimates for η_s/η_1. It is not clear that the fact that the isotropic curve and the bilaminate limits possess a center of symmetry requires that the curves for intermediate a/b do as well. The present estimates for η_n/η_1 and η_s/η_1 do show this symmetry, supporting her conjecture.

6. Aligned isotropic inclusions in an anisotropic host

In many natural examples, the matrix is intrinsically anisotropic. Deformed conglomerates (Treagus and Treagus, 2002) may have well-developed matrix foliation. Foliation will tend to wrap around clasts, and be weaker in strain shadows and stronger in strain concentrations, so an assumption of a homogeneous matrix is an approximation. The anisotropic matrix case allows a connection to be made between the bilaminate limit, in which one layer component is anisotropic, as in Bayly's (1970) study, and the clast/matrix geometry. Rocks may often have anisotropic components, such as anisotropic clasts of metamorphic rock. The detailed results developed here are limited to cases in which the principal axes of anisotropy of clasts and matrix coincide with the shape axes, but the solution in Appendix A covers the others.

When either matrix or clasts are anisotropic and the clasts are elliptical, the question arises as to the relative contributions of the intrinsic anisotropy of one or both components and the clast shape to the bulk anisotropy. The low values of m estimated for shape anisotropy suggest that it may not always be the more significant contributor. Here, the case of isotropic clasts in anisotropic matrix is considered.

Returning to Eq. (9), we may write the SCA relations for the general case, in which $\eta_n^* \neq \eta_s^*$. Since the matrix is anisotropic, with principal viscosities $\eta_n^{(1)}$ and $\eta_s^{(1)}$, the question arises as to which to use in the normalization. We choose $\eta_n^{(1)}$, so that

$$\beta_n = \frac{\eta_n}{\eta_n^{(1)}} \qquad R = \frac{\eta_2}{\eta_n^{(1)}} \tag{33}$$

and introduce the additional parameter

$$m_1 = \frac{\eta_n^{(1)}}{\eta_s^{(1)}} \tag{34}$$

The first relation in Eq. (19) then remains the same and

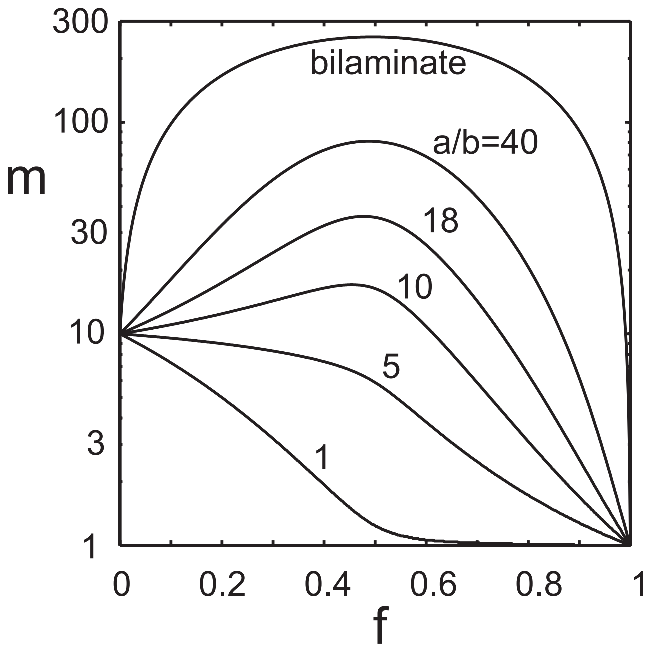

Fig. 4. Principal viscosity ratio, $m = \eta_n/\eta_s$, for $R = 100$, $m_1 = 10$, and $a/b = 1$, 5, 18, 30, and ∞.

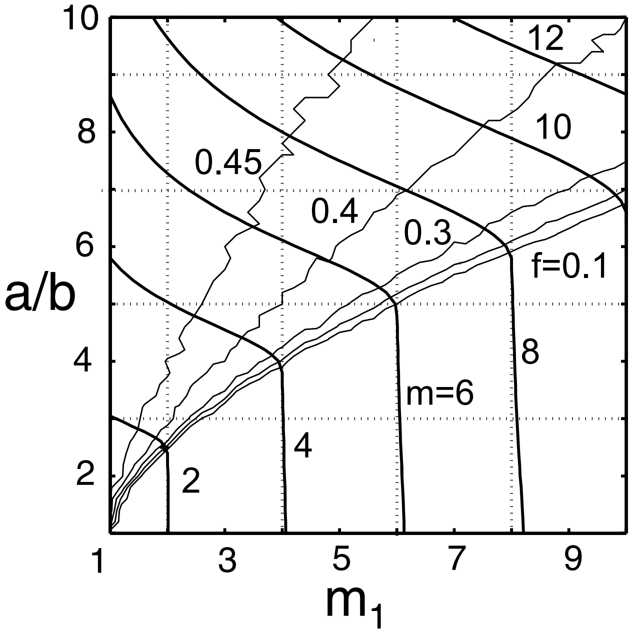

Fig. 5. Contours of maximum composite anisotropy, *m*, and the clast fraction, *f*, at which it occurs in a matrix with $R_G = 100$ in *a*/*b*, m_1-space.

the second becomes

$$m_1\beta_n^2 + \left\{\nu\sqrt{m}\left[f + (1-f)m_1R\right] - m\left[(1-f) + fm_1R\right]\right\} \times\beta_n - \nu m\sqrt{m}R = 0 \quad (35)$$

The method of solution for *m* and β_n is the same as that used earlier.

For a moderately anisotropic matrix, $m_1 = 10$, and much stiffer clasts, $R = 100$, adding equant clasts, $a/b = 1$, diminishes the bulk anisotropy (Fig. 4), effectively eliminating it at $f = 0.5$. A very large clast aspect ratio is needed to approach the bilaminate limit, as in the case of an isotropic matrix.

An alternative characterization of the matrix/clast contrast

$$R_G = \frac{\eta_2}{\sqrt{\eta_n^{(1)}\eta_s^{(1)}}} = R\sqrt{m_1} \quad (36)$$

is fixed to give the result shown in Fig. 5. A sharp transition occurs between composites dominated by matrix anisotropy to those dominated by shape anisotropy (Fig. 5). The value of *f* at which the maximum occurs is given as a second set of contours. In the lower region, the maximum occurs at small clast fraction and the anisotropy is dominated by that of the matrix; in the upper region, the maximum occurs at a clast fraction approaching $f = 0.5$, and bulk anisotropy is dominated by shape anisotropy. At the transition, the two contribute sub-equally.

7. Summary and conclusions

An auxiliary solution for an elliptical inclusion in an anisotropic viscous host is used to obtain estimates for the bulk properties for a dispersion of aligned, isotropic viscous clasts in an isotropic viscous matrix. In the spirit of previous studies, the host is treated as having the postulated bulk constitutive relations of the composite. Treagus (2003) instead treats the host in the auxiliary problem as isotropic. The two methods give substantially different estimates, with the present method giving weaker anisotropy, as measured by the principal viscosity ratio $m = \eta_n/\eta_s$. Study of the special case of a dispersion of rigid elliptical inclusions suggests that the present estimates are to be preferred.

The tendency for the strength of anisotropy to be markedly reduced when the composite differs from the bilaminate limit may explain a reduction in small-scale folding in gneissic rocks in which mineralogically distinct domains are of lenticular form. Treagus (2003) evaluates values of *m* from independent estimates of relative component viscosity. These values apply to an ideal bilaminate, but estimates for clast-matrix rocks using the same relative viscosities would be smaller by the present method. Folding instability of such rocks would be extremely weak.

The method allows the more general case of isotropic inclusions in an intrinsically anisotropic matrix to be treated, and a transition in behavior occurs between clast concentrations in which the bulk anisotropy is dominated by the intrinsic anisotropy and those in which it is dominated by shape anisotropy.

A general solution for anisotropic inclusion and host, with principal axes of anisotropy oblique to the shape axes of the inclusion, is given.

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Appendix A. Solution for elliptical anisotropic inclusion in an anisotropic matrix

This solution is obtained from that for an elliptical cavity in an anisotropic elastic body in plane strain (Lekhnitskii, 1963). The procedure for replacing the cavity with an elliptical inclusion (Eshelby, 1957) guarantees continuity of tractions and velocity at the interface. Specialization to incompressible host and inclusion, and replacement of infinitesimal elastic strain with the rate of deformation transforms the elastic solution to that used here.

Coordinate axes x and y are taken parallel to the principal axes of the elliptical inclusion. Constitutive relations specialized for plane flow are, for the host

$$D_{xx} = \beta_{11}\left(\sigma_{xx} - \sigma_{yy}\right) + \beta_{16}\sigma_{xy} \qquad D_{yy} = -D_{xx}$$
$$D_{xy} = \beta_{16}\left(\sigma_{xx} - \sigma_{yy}\right) + \frac{\beta_{66}}{2}\sigma_{xy} \tag{A1}$$

The orthotropic anisotropy of the inclusion, as well as that of the host may both have principal axes oblique to the 'shape' axes of the elliptical inclusion. They have the same form as in Eq. (A1), but with all quantities with an asterisk. If δ is the angle between the principal axes of anisotropy in the host and the ellipse axes, and η_n and η_s are the principal viscosities in foliation-parallel shortening/extension and shear, respectively, then

$$\beta_{11} = \frac{1}{8\eta_n}[(1+m) + (1-m)\cos 4\delta]$$
$$\frac{\beta_{66}}{2} = \frac{1}{4\eta_n}[(1+m) - (1-m)\cos 4\delta] \tag{A2}$$
$$\beta_{16} = \frac{1}{4\eta_n}(1-m)\sin 4\delta$$

with equivalent expressions for the inclusion. The solution for the stress and velocity mediating between the homogeneous far field stress and that in the inclusion depends on the roots, μ_k, of the characteristic equation associated with the fourth-order equation in the Airy stress function obtained by substituting the constitutive relations into the compatibility equation

$$\beta_{11}\mu^4 + 2\beta_{16}\mu^3 + (\beta_{66} - 2\beta_{11})\mu^2 + \beta_{11} = 0 \tag{A3}$$

The solutions are

$$\mu_1 = \alpha_1 + i\beta_1, \;\; \mu_2 = \alpha_2 + i\beta_2 \tag{A4}$$

and their complex conjugates, where

$$\alpha_1 = \frac{2\sqrt{(m-1)}(\sqrt{m} - \sqrt{(m-1)}\cos 2\delta)\sin 2\delta}{[(m+1) - (m-1)\cos 4\delta]}$$
$$\alpha_2 = \frac{-2\sqrt{(m-1)}(\sqrt{m} + \sqrt{(m-1)}\cos 2\delta)\sin 2\delta}{[(m+1) - (m-1)\cos 4\delta]}$$
$$\beta_1 = \frac{2(\sqrt{m} - \sqrt{(m-1)}\cos 2\delta)}{[(m+1) - (m-1)\cos 4\delta]} \tag{A5}$$
$$\beta_2 = \frac{2(\sqrt{m} + \sqrt{(m-1)}\cos 2\delta)}{[(m+1) - (m-1)\cos 4\delta]}$$

The boundary conditions for continuity of velocity at the host inclusion interface are substantially simplified for incompressible media. Relations important in simplifying the velocity boundary conditions, which were initially derived for compressible elastic solids are

$$\alpha_1\alpha_2 - \beta_1\beta_2 = -1 \qquad \alpha_1\beta_2 + \alpha_2\beta_1 = 0$$
$$\left(\alpha_1^2 + \beta_1^2\right)\left(\alpha_2^2 + \beta_2^2\right) = 1 \tag{A6}$$

The velocity boundary conditions then yield the following relations between the components of homogeneous stress in the inclusion, denoted by an asterisk, as in s_{xx}^* and the far field stress, denoted by a superscript, as in s_{xx}^o. These are

$$\sigma^* = \sigma^o + \frac{\left(\frac{b}{a} - \frac{a}{b}\right)}{\left(\frac{b}{a} + \frac{a}{b}\right)}\left(s_{xx}^* - s_{xx}^o\right)$$
$$\times\left[2\left(\frac{b}{a} + \frac{a}{b}\right)^{-1}\left(\beta_1 + \beta_2\right)\beta_{11} + 2\beta_{11}^*\right]s_{xx}^*$$
$$+\left[\left(\alpha_1 + \alpha_2\right)\beta_{11} - \beta_{16}^*\right]s_{xy}^*$$
$$= \left[2\left(\frac{b}{a} + \frac{a}{b}\right)^{-1}\left(\beta_1 + \beta_2\right)\beta_{11} + 2\beta_{11}\right]s_{xx}^o$$
$$+\left[\left(\alpha_1 + \alpha_2\right)\beta_{11} - \beta_{16}\right]s_{xy}^o \tag{A7}$$
$$-2\left[\left(\alpha_1 + \alpha_2\right)\beta_{11} - \left(\beta_{16} + \beta_{16}^*\right)\right]s_{xx}^*$$
$$+\left[\left(\frac{b}{a} + \frac{a}{b}\right)\left(\beta_1 + \beta_2\right)\beta_{11} + \beta_{66}^*\right]s_{xy}^*$$
$$= -2\left[\left(\alpha_1 + \alpha_2\right)\beta_{11} - 2\beta_{16}\right]s_{xx}^o$$
$$+\left[\left(\frac{b}{a} + \frac{a}{b}\right)\left(\beta_1 + \beta_2\right)\beta_{11} + \beta_{66}\right]s_{xy}^o$$

Having solved these relations for the stresses in the host, the vorticity in the inclusion, ω^*, is related to the far-field value, ω^{o}, by

$$2\omega^{*}\left[\frac{\left(\frac{b}{a}-\frac{a}{b}\right)}{\left(\frac{b}{a}+\frac{a}{b}\right)}\right](\beta_1+\beta_2)\beta_{11}s^{*}_{xy} - 2[(\alpha_1+\alpha_2)\beta_{11}-\beta_{16}]\sigma^{*} = 2\omega^{o}\left[\frac{\left(\frac{b}{a}-\frac{a}{b}\right)}{\left(\frac{b}{a}+\frac{a}{b}\right)}\right](\beta_1+\beta_2)\beta_{11}s^{o}_{xy} - 2[(\alpha_1+\alpha_2)\beta_{11}-\beta_{16}]\sigma^{o} \quad \text{(A8)}$$

In the present application, the inclusions are aligned along the principal axes of anisotropy in the host, so that $\delta = 0$. Quantities used in the general conditions obtained from the velocity boundary conditions reduce to

$$\alpha_1+\alpha_2 = 0 \qquad \beta_1+\beta_2 = 2\sqrt{m} \quad \text{(A9)}$$

Excluding the first relation in Eq. (A7), which continues to hold, the remaining two conditions reduce to

$$\left[2\left(\frac{b}{a}+\frac{a}{b}\right)^{-1}\sqrt{m}\beta_{11}+\beta^{*}_{11}\right]s^{*}_{xx} = \left[2\left(\frac{b}{a}+\frac{a}{b}\right)^{-1}\sqrt{m}\beta_{11}+\beta_{11}\right]s^{o}_{xx}$$
$$\left[2\left(\frac{b}{a}+\frac{a}{b}\right)\sqrt{m}\beta_{11}+\beta^{*}_{66}\right]s^{*}_{xy} = \left[2\left(\frac{b}{a}+\frac{a}{b}\right)\sqrt{m}\beta_{11}+\beta_{66}\right]s^{o}_{xy} \quad \text{(A10)}$$

These are the relations used in the text.

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